Discrete Maximum Principle in the Family of Mimetic Finite Difference Methods

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The maximum principle is one of the most important properties of solutions of partial differential equations. To preserve this property on a discrete level is a challenging problem in many applications. The analog of the maximum principle for a discrete system is referred to as the DMP. In our research we investigate the conditions sufficient to ensure that the family of the MFD method contains a subfamily that satisfies DMP.

Standard discretization methods, such as Finite Element (FE) or Finite Volume (FV) methods, guarantee the discrete maximum principle (DMP) only under severe limitations on either the computational mesh or the material properties. For isotropic materials, FE methods limit the angles of admissible simplices. FV methods require centroidal Voronoi meshes, where the line connecting the centers of two neighboring control volumes is orthogonal to its common face. For anisotropic materials (shale deposits in subsurface and magnetized plasma), the above requirements must be reformulated using the material-dependent anisotropic metric. A mesh that satisfies such requirements may exist only for academic problems with simple geometry.

In more general settings when the computational mesh is distorted, or the problem coefficients are anisotropic, these methods may produce unphysical numerical solutions. A violation of the DMP leads to numerical instabilities, such as "overshoots" and "undershoots," and to unphysical fluxes such as heat flow from a cold material to a hot one.

Recently developed nonlinear FV methods [1] can handle general types of meshes and material properties, but the "price" for this exceptional capability is the nonlinear nature of the methods, even in the case of linear problems. The nonlinearity of these methods results in an increase of computational cost of 5–20 times compared to its linear analogs. Therefore linear methods that satisfy the DMP principle are very valuable.

The family of MFD methods consists of linear discretization methods that were designed to discretize diffusion-type problems with a full-diffusion tensor on general polygonal/polyhedral meshes [2]. In particular, MFD methods can be applied to the diffusion-type problems written in the mixed form as follows:

$$\mathbf{u} = -K\mathrm{grad}\rho \quad \text{and} \quad \mathrm{div}\mathbf{u} = Q.$$

where ρ is an unknown scalar variable referred to as pressure, $\mathbf{u}$ is an unknown flux vector field, K is a diffusion tensor, and Q is a source function.

The approximation of the divergence operator is similar in many approaches and is based on the divergence theorem. The key element of the MFD method is the approximation of the gradient operator, which is not unique. This fact allows one to locally tune the discretization method to adapt it to media properties and cell geometry. In the family of MFD methods, the definition of the discrete gradient operator depends on the set parameters, and a particular choice of these parameters defines a member in the family. The discrete gradient operator can be defined locally and the number of local parameters depends on a cell type. Here we present the dimension of local parameter spaces for several cell types, shown in Table 1.

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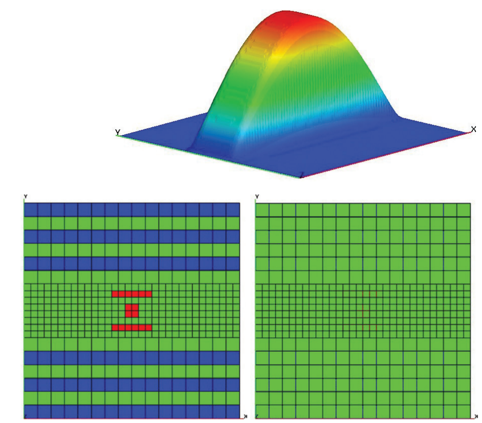
Table 1. Number of discretization parameters for different cell types.

Cell	Number of Parameters
	1
	3
	6
	6

Different formulations of a DMP are possible since they can be derived from different formulations of the continuous maximum principle. One of the possible formulations of a DMP is based on the non-negativity of the inverse of the stiffness matrix. The discretization method that satisfies this property is referred to as a *monotone* method. An effective way to ensure monotonicity is to construct a discretization method that resolves into the algebraic system with an M-matrix. We choose several types of cells of practical importance, including simplices, parallelograms, parallelepipeds, and cells in adaptive mesh refinement (AMR) meshes, in order to formulate the limits for when a monotone subfamily exists (see [3]). We propose the choice of discretization parameters that lead to a monotone discretization method for a particular mesh type. In Fig. 1 we present the comparison between the standard MFD method described in [2] and the monotone MFD method for the problem defined on AMR mesh. The numerical results show that the monotone MFD method satisfies the DMP and does not compromise the accuracy of the method, while the standard approach fails.



Standard MFD method

Monotone MFD method

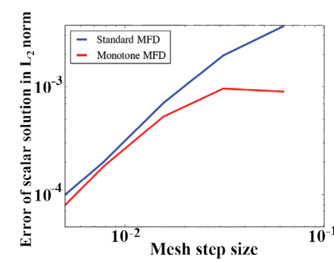


Fig. 1. (Top) The profile of the solution on the AMR grid for the problem with an anisotropic diffusion tensor and a heterogeneous source function. (Center) The standard MFD method produces the solution with large subdomains of "overshoots" (red) and "undershoots" (blue), while the monotone MFD method produces an oscillation-free solution. (Bottom) The monotone MFD method provides the solution that not only satisfies the DMP, but also is more accurate on coarser grids.

[1] Lipnikov, K., et al., *J Comp Phys* **228,** 703 (2009).
[2] Brezzi, F., et al., *Math Model Meth Appl Sci* **15,** 1533 (2005).
[3] Lipnikov, K., et al., "Analysis of the Monotonicity Conditions in the Mimetic Finite Difference Method for Elliptic Problems," LA-UR-10-05056, *J Comp Phys,* submitted (2010).

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